Lecture 3 - Snarks

Theorem 1 (Vizing)**.** *Every simple graph satisfies*

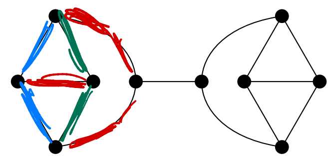
$$\chi'(G) \le \Delta(G) + 1.$$

Can you find graphs G such that $\Delta(G) = 3$ and $\chi'(G) = 4$?

Goal is to define snarks as cubic graphs that are not 3-edge-colorable. Name inspired by "real" snarks that are some elusive, rare creatures in the sea from the book *The Hunting of the Snarks* by Lewis Carrol.

Here we try to investigate what properties snarks should have and then we define them later.

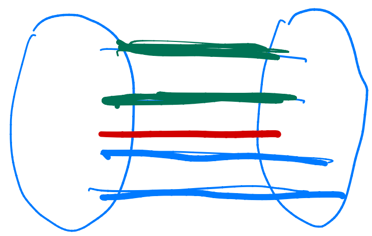
1: What is chromatic index of the following graph?



$\chi'(G) = 4$

Lemma 2 (Blanuša)**.** *Let $c : E(G) \to \{1, 2, 3\}$ be a 3-edge-coloring of a cubic graph G and let C be an edge-cut. Denote by c_i the number of edges of C colored i. Then,*

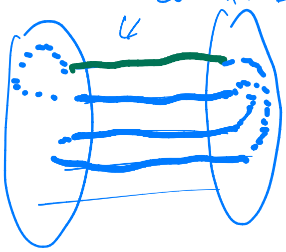
$$c_1 \equiv c_2 \equiv c_3 \pmod 2.$$

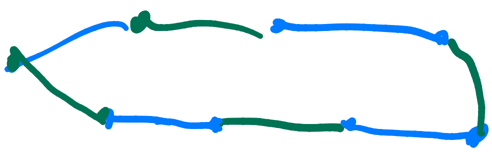


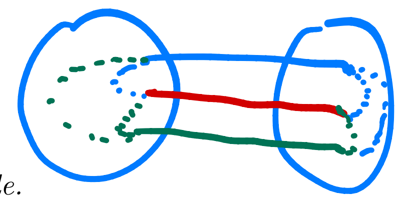
$c_1 = 2$ $c_2 = 2$ $c_3 = 1$

2: Prove the lemma. Hint: Consider Kempe chains.

LOOK AT KEMPE CHAIN

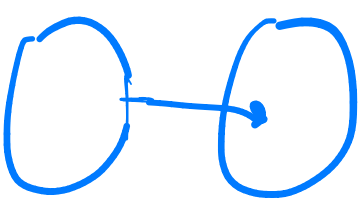






Proposition 3. *If a cubic graph has a bridge then it is not 3-edge-colorable.*

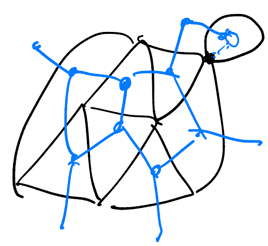
3: Prove the proposition.



$c_1 = 1$

$c_2 = 0$

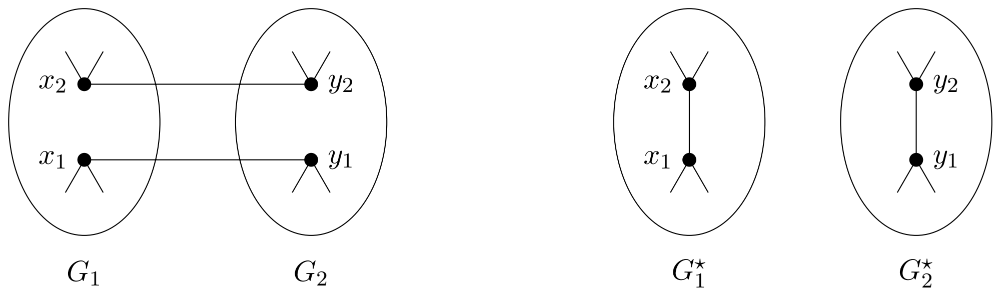
$c_3 = 0$



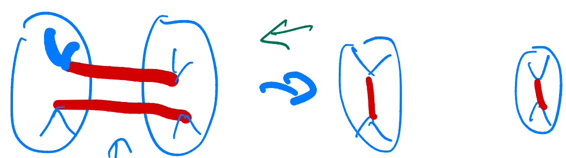
Bridges corresponds to loops in the dual graph, and we do not consider them in vertex colorings of planar graphs. So, we do not like our snarks to have bridges, as we want to relate the snarks with the Four Color Theorem.

Proposition 4. *Let G be a cubic graphs with 2-edge-cut $\{x_1y_1, x_2y_2\}$ and let G_1 and G_2 the components of $G - x_1y_1 - x_2y_2$ as it is depicted below and finally let $G_1^* = G_1 + x_1x_2$ and $G_2^* = G_2 + y_1y_2$. Then $\chi'(G) = 4$ if and only if $\chi'(G_1^\star) = 4$ or $\chi'(G_2^\star) = 4$.*

x_1x_2 & y_1y_2 ARE NOT EDGES.

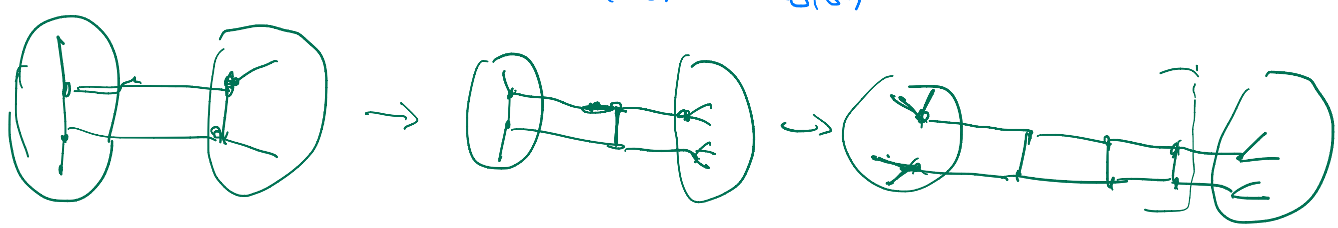


4: Prove the proposition.



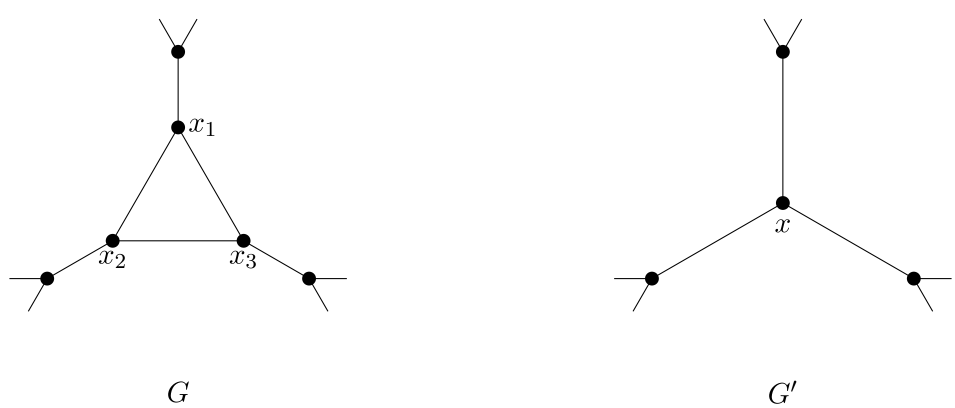
$\chi'(G) = 3 \Rightarrow \chi(G_1^*) = \chi'(G_2^*) = 3$

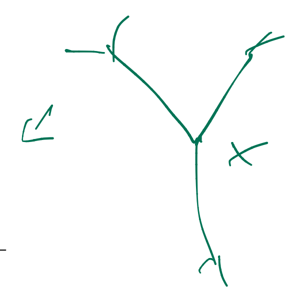
$\chi(G_1^*) = \chi(G_2^*)$.. MAKE $c(x_1x_2) = c(y_1y_2)$ MUST BE SAME DUE TO BUNŠA



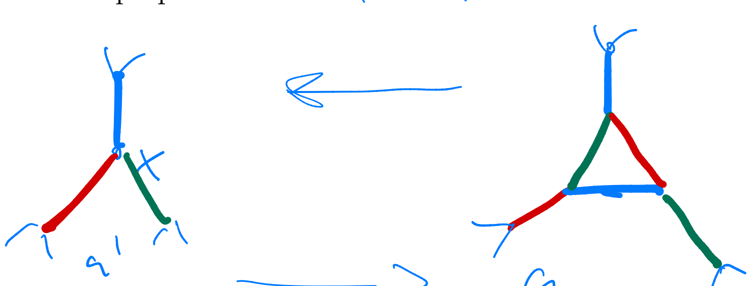
Proposition 5. *Let G be a cubic graphs with a 3-cycle $x_1x_2x_3$, and let G' be the cubic graph obtained from G by contracting this 3-cycle. Then, $\chi'(G) = 4$ if and only if $\chi'(G') = 4$.*

This is called Δ/Y operation for obvious reason.



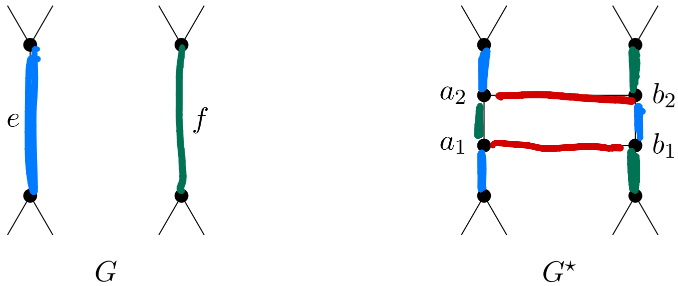


5: Prove the proposition. $\chi'(G) = 3 \Leftrightarrow \chi'(G) = 3$



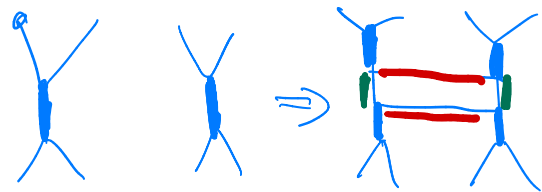
Now we deal with 4-cycles.

Proposition 6. *Let G be a cubic graphs and e and f two edges of G. Subdivide twice e by vertices a_1 and a_2 and also subdivide twice f by vertices b_1 and b_2, and connect a_1 by b_1 and a_2 by b_2. Denote the resulting graph by G^*. If $\chi(G^*) = 4$, then $\chi(G) = 4$.*



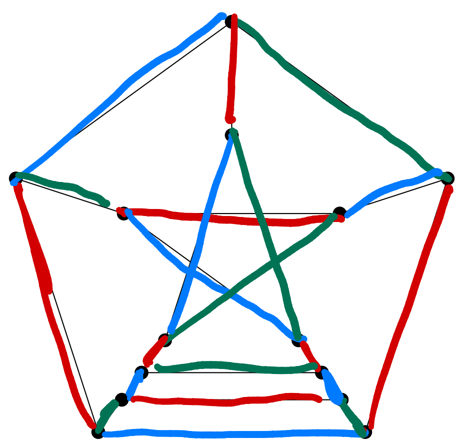
2 CASES TO 3-EDGE COLOR G

6: Prove the proposition.

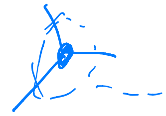


Note that in the previous proposition does not hold "if and only if", i.e., it can happen that $\chi(G) = 4$ but $\chi(G^*) = 3$. This can be easily seen by taking G to be the Petersen graph.

7: Show that the following graph is 3-edge-colorable.



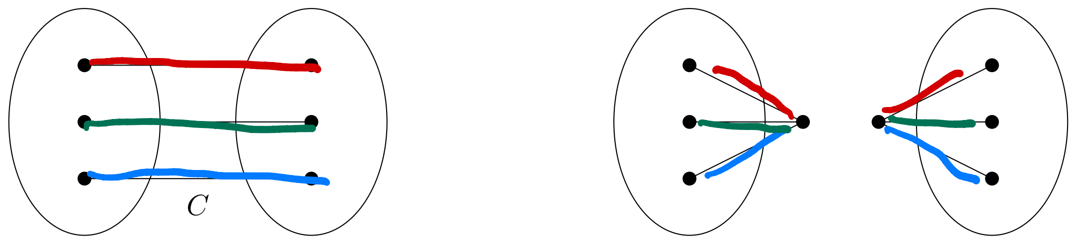
← 3-EDGE COLORING



ALWAYS A CUT

For next, let us consider nontrivial 3-edge-cuts.

Proposition 7. *Let G be a cubic graph with nontrivial 3-edge-cut C, and let $G - C$ has the components G_1 and G_2 and $G_1^* = G/G_2$ and $G_2^* = G/G_1$. Then $\chi'(G) = 4$ if and only if $\chi'(G_1^*) = 4$ or $\chi'(G_2^*) = 4$.*

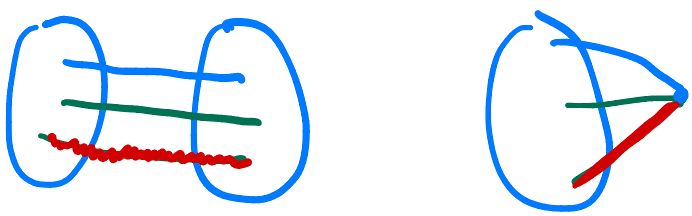


G

3 $\longleftarrow$ 3

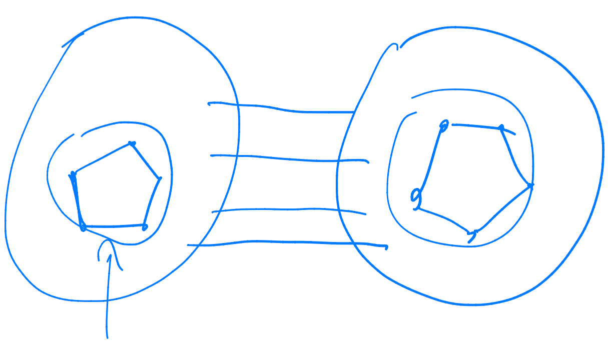
3-EDGE COLORINGS ?

8: Prove the proposition.

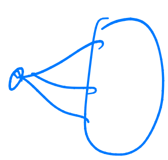


Above propositions, somehow tell that the presence of 3- and 4-cycles is not the real reason while should a cubic graph be non-3-edge-colorable, and similarly holds for 2-edge-cuts and 3-edge-cuts. So we do not like snarks to have such objects. Following the line of this reasoning with few more similar claims, which we omit here, we conclude with a definition of the snarks as: ***snarks*** *are cyclically 4-edge-connected cubic graphs of girth at least five with chromatic index 4.*

CYCLICALLY 4-EDGE CONNECTED:



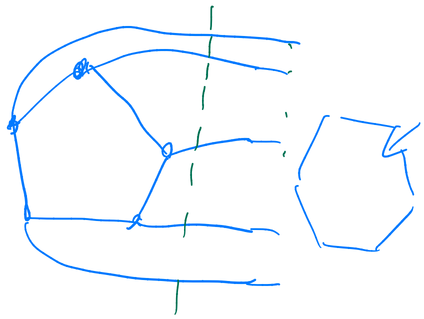
IF C IS EDGE CUT ST. IN G−C BOTH COMPONENTS CONTAIN A CYCLE THEN $|C| \geq 4$



Proposition 8. *The Petersen graph is the smallest snark.*

Proof. Suppose that G is a snark on ≤ 9 vertices or it is a snark on 10 vertices distinct from the Petersen graph.

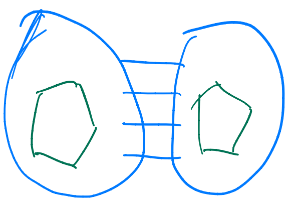
9: Show that G has a cyclic edge-cut. Hint: start with shortest cycle.



AT LEAST 5 CYCLE

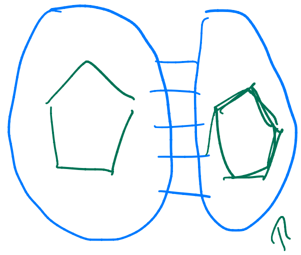
$\Rightarrow \geq 10$ VERTICES

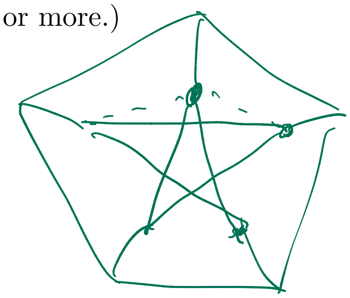
10: Suppose now that $Z = \{x_0y_0, x_1y_1, \ldots, x_{k-1}y_{k-1}\}$ a smallest cyclic edge-cut of G with all x_i's on one side and finish the proof. (Note: definition guarantees only 4-edge cut or more.)



5 5 X

4 × 2-VTX
1 × 3-VTX





□

Until 1975, there were known only 5 snarks: the Peteresen graph, the two Blanuša snarks on 18 vertices depicted below. The Szekeres snark on 50 vertices and the Descartes snark on 210 vertices. Then, Isaacs came with two constructions of infinitely many snarks.

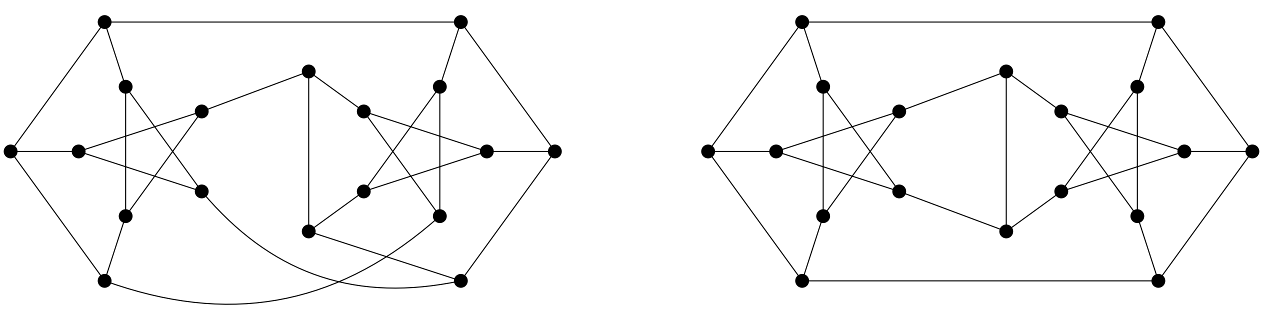
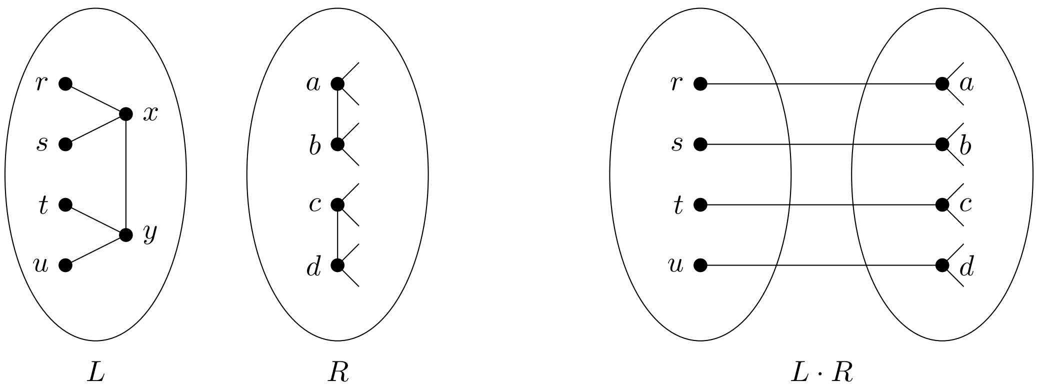


Figure 1: Blanuša snarks.

Dot product. A *dot product* of two cubic graphs L and R, denoted by $L \cdot R$, is defined as follows:

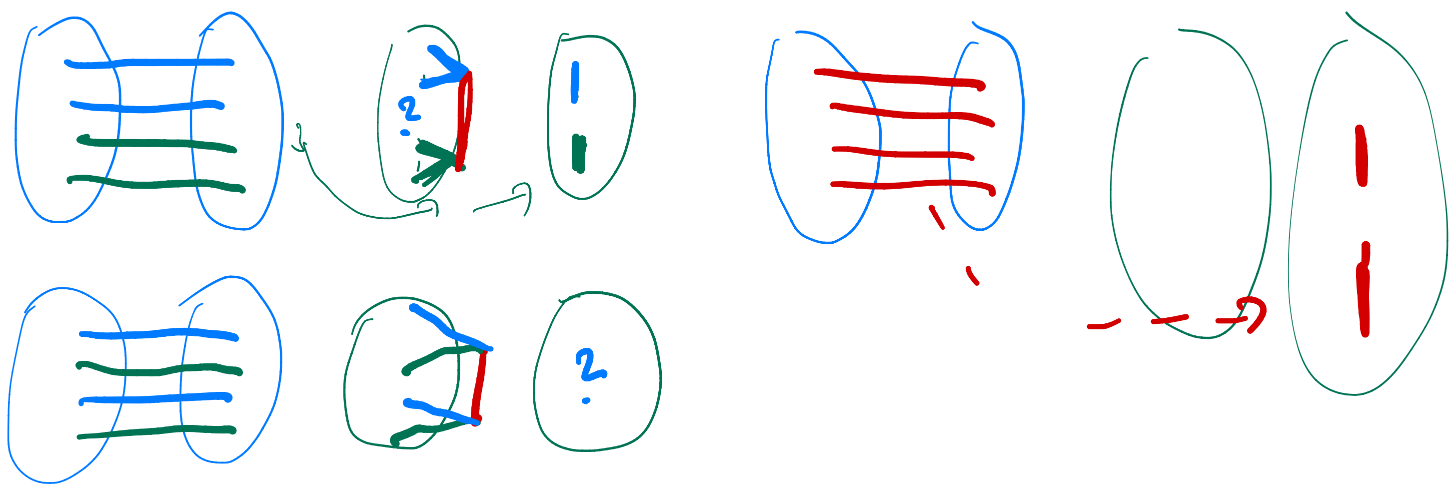
1. remove adjacent vertices x and y of L, where $N(x) = \{r, s, y\}$ and $N(y) = \{t, u, x\}$;
2. remove any two independent edges ab and cd from R;
3. connect r with a, s with b, t with c, and u with d.



Following claim enable us to construct infinitely many snarks.

Proposition 9. *If L and R are 4-edge-chromatic cubic graphs, then $L \cdot R$ is 4-edge-chromatic.*

11: Prove the proposition. Hint: Suppose for contradiction that $L \cdot R$ is 3-edge-colorable.

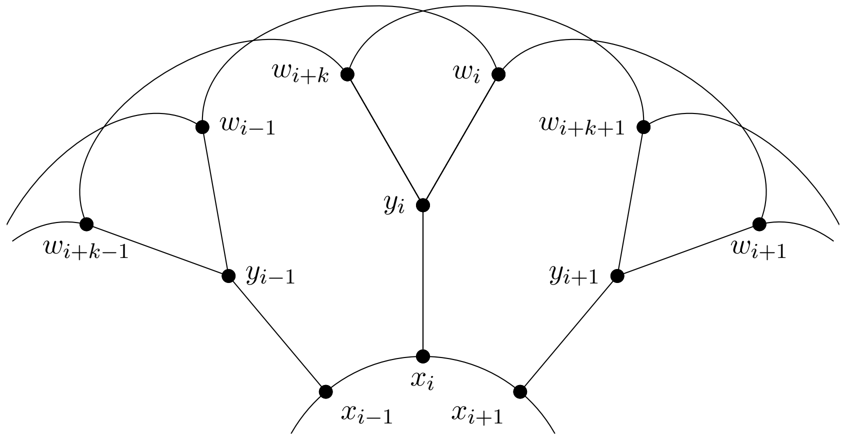


Flower snarks. Let $k \geq 3$ be an odd. We define the graph F_k in the following way:

1. $V(F_k) = \{x_i, y_i : i \in I(k)\} \cup \{w_i : i \in I(2k)\}$,
2. $E(F_k) = \{x_i x_{i+1}, x_i y_i, w_j w_{j+1} : i \in I(k), j \in I(2k)\} \cup \{y_i w_i, y_i w_{i+k} : i \in I(k), i + k \in I(2k)\}$,

where we take indices at u's and v's modulo k, and at w's modulo $2k$. See Figure 2 for a local sketch of the construction.

Proposition 10. *For every odd $k \geq 5$, the graph F_k is a snark.*

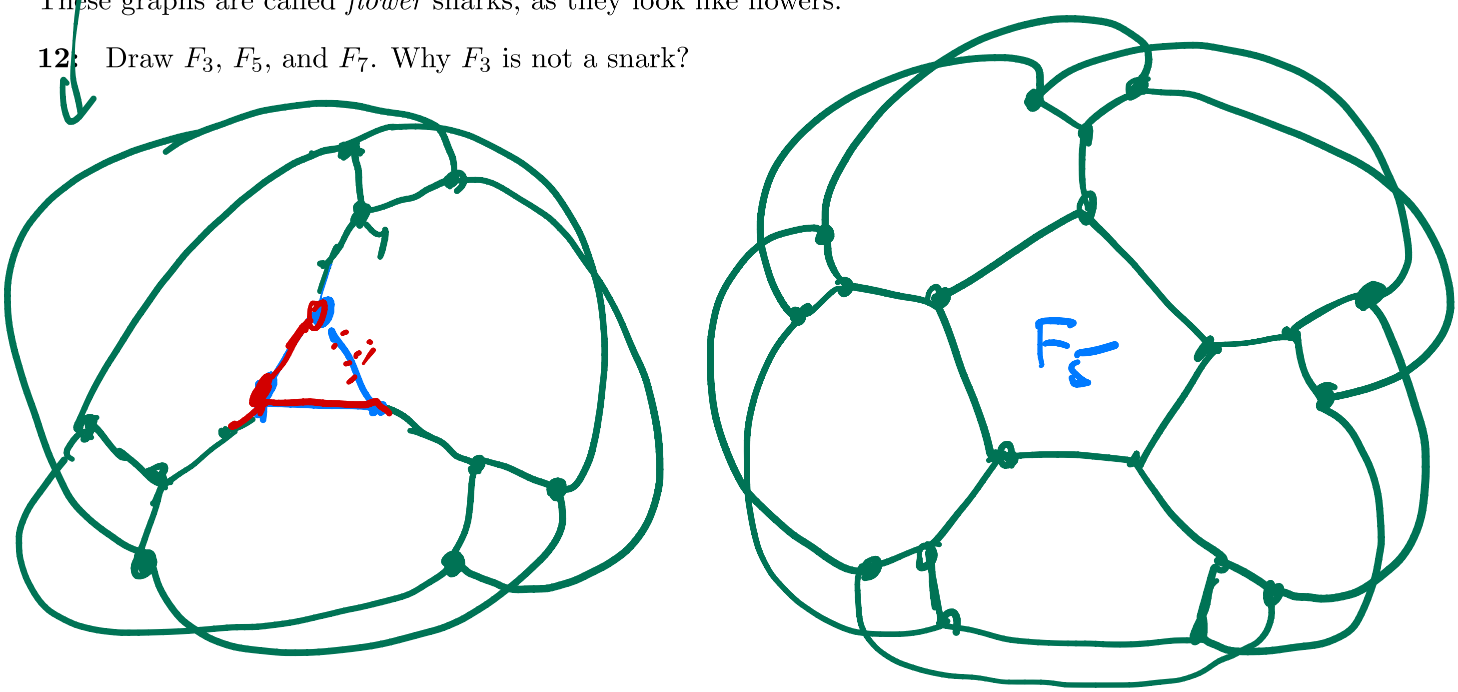


PETERSEN

Figure 2: Local structure of a flower snark.

These graphs are called *flower* snarks, as they look like flowers.

12: Draw F_3, F_5, and F_7. Why F_3 is not a snark?



1 Some conjectures related to the snarks

Most central conjecture regarding edge-coloring of cubic graphs is the Tutte conjecture. Almost 20 years ago, a proof was announced by Sanders, Robertson, Seymour and Thomas, but their proof never appeared. Note that this conjecture can be restated as every snark contains a subdivision of the Petersen graph.

Conjecture 11 (Tutte)**.** *Every Petersen minor-free bridgeless cubic graph is 3-edge-colorable.*

$\subset$ SUBDIVISION

13: Find subdivisions of Petersen in F_3, F_5, and F_7.

In what follows, we consider several central conjectures in this area. The first one is by Fulkerson from 1971.

Conjecture 12 (Fulkerson)**.** *Let G be a bridgeless cubic graph. If a graph H is obtained by doubling every edge in a simple G, then it is 6-edge-colorable.*

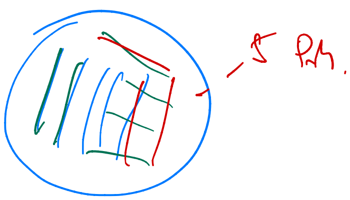
14: Why is the above conjecture trivial for 3-edge-colorbale graphs? (i.e. not snarks)

... Δ, Δ+1

Δ ENOUGH

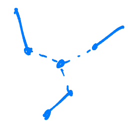
A slightly weaker conjecture but still popular is a conjecture that edges of a cubic graph can be covered by 5 perfect matchings.

15: Why is the 5 perfect matching conjecture implied by the Fulkerson's conjecture?



TAKE 6-EDGE COLORING & FORGET 1 COLOR

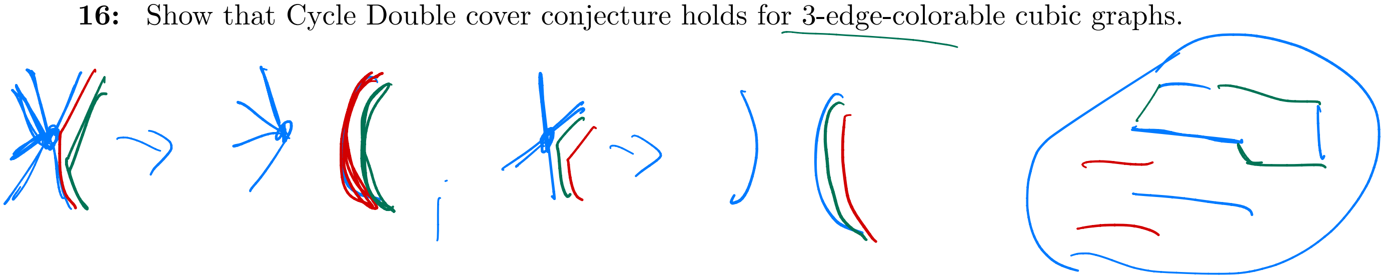
Δ=3 (Δ=6)



Very popular conjecture is the following one, which was independently proposed by Szekeres and Seymour, it is known under the name *Cycle Double Conjecture*. At first glance, it looks that this conjecture has nothing to do with edge-colorings of cubic graphs but let us remark that the conjecture holds in general if it is true for cubic graphs. The conjecture has several stronger versions and one of them is of topological flavor and claims that every 2-connected graphs can be embedded on a surface such that each face is a cycle.

Conjecture 13 (Szekeres, Seymour)**.** *Edges of every 2-edge-connected graphs can be covered by cycles (possible some of them appear twice), such that each edge belongs to precisely two cycles.*

16: Show that Cycle Double cover conjecture holds for 3-edge-colorable cubic graphs.



Another very attractive (but perhaps also difficult) conjecture is the following one of Jaeger. It implies both the Fulkerson Conjecture and the Cycle Double Conjecture.

Conjecture 14 (Jaeger)**.** *Every cubic bridgeless graph G can be edge-colored with colors that represent the edges of the Petersen graph such that any three mutually adjacent edges of G are colored with colors that represent three mutually adjacent edges of the Petersen graph.*

17: Show that Jaeger's conjecture is true for 3-edge-colorable cubic graphs.